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COMPACT COMPLEX THREEFOLDS OF CLASS L ASSOCIATED TO POLYNOMIAL AUTOMORPHISMS OF \mathbb{C}^3

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Abstract

We construct new families of non-Kähler compact complex threefolds belonging to Kato's Class L . The construction uses certain polynomial automorphisms of \mathbb{C}^3 . We also study basic properties of our manifolds.

1. Introduction and Construction

We start by recalling the definition of Kato's Class L , see [6]. Let

$$V := \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{P}_3(\mathbb{C}) \mid |z_0|^2 + |z_1|^2 > |z_2|^2 + |z_3|^2 \right\}.$$

A (not necessarily compact) irreducible complex space X is said to be of *Class L* , if and only if X contains an open subset biholomorphic to U . M. Kato studied several interesting (smooth) examples and properties of Class L spaces, see [6] and its references. The aim of the present paper is to construct new families of examples of non-Kähler class L spaces and threefolds with fundamental group \mathbb{Z} . All previously known examples of such manifolds were quotients of the complement of two disjoint linear rational curves in $\mathbb{P}_3(\mathbb{C})$, which all admit projective structures (see [7]).

In order to simplify the approach, we give the construction and study the properties in a special case. In the last section we give all quadratic automorphisms of \mathbb{C}^3 which allow exactly the same arguments.

Consider the following polynomial automorphism H of \mathbb{C}^3

$$H(x, y, z) = \begin{pmatrix} x^2 + cy^2 + z \\ y^2 + x \\ y \end{pmatrix},$$

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the constant $c \in \mathbb{C}$ being chosen arbitrarily. The inverse automorphism is given by

$$H^{-1}(x, y, z) = \begin{pmatrix} y - z^2 \\ z \\ x - (y - z^2)^2 - cz^2 \end{pmatrix}.$$

We extend H and H^{-1} to $\mathbb{P}_3 = \{[x : y : z : t]\}$ and, keeping the same notations, we get:

$$H[x : y : z : t] = [x^2 + cy^2 + zt : y^2 + xt : yt : t^2]$$

$$H^{-1}[x : y : z : t] = [yt^3 - z^2t^2 : zt^3 : xt^3 - (yt - z^2)^2 - cz^2t^2 : t^4].$$

The set of indeterminacy of H is given by

$$I^+ = \{x = y = t = 0\}$$

which is the point $[0 : 0 : 1 : 0]$, and that of H^{-1} equals to

$$I^- = \{z = t = 0\}$$

which is the projective line $\{[x : y : 0 : 0]\}$.

It is clear that $I^+ \cap I^- = \emptyset$, i.e. automorphism H is regular in the sense of Sibony [8].

Definition 1.1. Let M be a metric space and $f: M \rightarrow M$ a continuous mapping. A compact subset $K \subset M$ is called attractor for f if there is an open neighbourhood $V \subset M$ of K such that $f(V) \Subset V$ and $\bigcap_{n \geq 0} f^n(V) = K$.

We have the following

Proposition 1.2 (see [8]). *Let f be a regular polynomial automorphism of \mathbb{C}^k of algebraic degree $d \geq 2$ considered as a birational map of $\mathbb{P}_k(\mathbb{C})$. Then the set of indeterminacy I^+ of f is an attractor for f^{-1} and, conversely, the set of indeterminacy I^- of f^{-1} is an attractor for f .*

The hyperplane at infinity minus the indeterminacy set $\{t = 0\} \setminus I^+$ is mapped by H to the projective line

$$X^+ := I^- = \{z = t = 0\} = \{[x : y : 0 : 0]\}$$

which is an attractor for H .

Let

$$U^+ := \left\{ w \in \mathbb{C}^3, \lim_{n \rightarrow \infty} H^n(w) \in X^+ \right\}$$

be the basin of attraction of H in \mathbb{C}^3 .

The automorphism H being algebraic, it follows that its graph $\text{gr}(H) \subset \mathbb{C}^3 \times \mathbb{C}^3$ has the property that its topological closure $W := \overline{\text{gr}(H)} \subset \mathbb{P}_3 \times \mathbb{P}_3$ is a (not necessarily smooth) projective variety. We have the following commutative diagram

$$\begin{array}{ccc} W \subset \mathbb{P}_3 \times \mathbb{P}_3 & & \\ \text{pr}_1 \downarrow & \searrow \text{pr}_2 & \\ \mathbb{P}_3 & \xrightarrow{H} & \mathbb{P}_3 \end{array}$$

where pr_i , $i = 1, 2$ are the projections to the \mathbb{P}_3 -factors. In the following we denote by p_i the restrictions of pr_i to W , $i = 1, 2$.

An easy calculation gives

$$W = \text{gr}(H) \cup D_1 \cup D_2$$

with

$$\begin{aligned} D_1 &:= \left\{ [z_0 : z_1 : z_2 : 0], [z_0^2 + cz_1^2 : z_1^2 : 0 : 0] \right\} \\ &\cup \left\{ [0 : 0 : 1 : 0], [y_0 : y_1 : 0 : 0] \right\}, \\ D_2 &:= \left\{ [0 : 0 : 1 : 0], [y_0 : y_1 : y_2 : 0] \right\}. \end{aligned}$$

For the intersection we have $l := D_1 \cap D_2 = \{[0 : 0 : 1 : 0], [y_0 : y_1 : 0 : 0]\}$.

Let $\varepsilon > 0$ and V_ε the neighbourhood of X^+ in \mathbb{P}_3 of the form

$$V_\varepsilon := \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{P}_3 \mid |z_2|^2 + |z_3|^2 < \varepsilon(|z_0|^2 + |z_1|^2) \right\}.$$

We note also $B_\varepsilon := \partial V_\varepsilon$ the boundary of V_ε .

Since X^+ is a attractor for H there is an $\varepsilon > 0$ such that

$$X^+ \subset V_\varepsilon \subset U^+ \cup \left(\{z_3 = 0\} \setminus \{[0 : 0 : 1 : 0]\} \right)$$

and

$$H(V_\varepsilon) \Subset V_\varepsilon.$$

The mapping $p_2^{-1} \circ \text{id} \circ p_1$ is biholomorphic onto its image in a open neighbourhood of $p_1^{-1}(B_\varepsilon) \subset W$ and therefore allows to identify holomorphically the two boundary components $p_1^{-1}(B_\varepsilon)$ and $p_2^{-1}(B_\varepsilon)$ of the complex space $A := p_2^{-1}(V_\varepsilon) \setminus \overline{p_1^{-1}(V_\varepsilon)}$. This identification gives rise to a compact complex space X with a global shell isomorphic B to B_ε , i.e. $(X \setminus B) = A$ is connected.

Let $N := \text{pr}_1^{-1}(V_\varepsilon)$ and $M := \text{pr}_2^{-1}(V_\varepsilon)$, i.e. $A = M \setminus N$. The following scheme illustrates the situation and the gluing up.

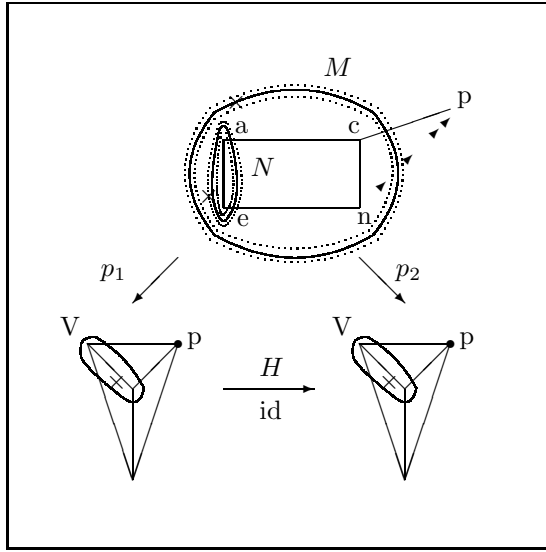


FIGURE 1. Gluing up.

The coordinates of the points are:

$$a = [1 : 0 : 0 : 0] [1 : 0 : 0 : 0]$$

$$n = [0 : 0 : 1 : 0] [0 : 1 : 0 : 0]$$

$$c = [0 : 0 : 1 : 0] [1 : 0 : 0 : 0]$$

$$p = [0 : 0 : 1 : 0] [0 : 0 : 1 : 0]$$

$$e = [0 : 1 : 0 : 0] [0 : 1 : 0 : 0].$$

By construction, the space X is the union of the quotient $U^+ / \langle H \rangle_{\mathbb{Z}}$ and of an irreducible divisor D with $\pi_1(D) = \mathbb{Z}$. Furthermore, the space X is singular exactly in the points corresponding to $l = D_1 \cap D_2$. Of course it is possible to resolve the singularities by a finite sequence of modifications of $\mathbb{P}_3 \times \mathbb{P}_3$ in order to obtain a smooth model of X . But since this process is not unique and does not change any of the properties discussed in the sequel, we shall work in what follows with the singular space. This space, and more generally all those that will be constructed

in the same way, provide examples of class L , since there are non-singular rational curves in $B \subset X$ admitting an open neighbourhood isomorphic to V .

Let us remark that in [7], M. Kato classified all compact manifolds of class L admitting a projective structure with fundamental group isomorphic to \mathbb{Z} . We shall see in what follows that our manifolds have this fundamental group. In view of Kato's result, it is easy to see that no smooth model of our spaces does admit any projective structure.

2. Properties

We shall use the same notations as in the previous paragraph.

Proposition 2.1. *The fundamental group $\pi_1(X)$ is isomorphic to \mathbb{Z} .*

Proof: Using the Mayer-Vietoris exact sequence, Van Kampen theorem and the fact that the space A is simply connected, the proof works like in [1, pp. 11–12]. \square

We continue by calculating the fundamental group of the attracting basin $U^+ \subset \mathbb{C}^3$ of H .

Proposition 2.2. $\pi_1(U^+) = 1$.

Proof: Recall that

$$V_\varepsilon = \left\{ \varepsilon(|z_0|^2 + |z_1|^2) > |z_2|^2 + |z_3|^2 \right\}$$

in homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ in \mathbb{P}_3 . It is evident that $V_\varepsilon \setminus \{z_3 = 0\}$ is simply connected.

Moreover, $U^+ = \cup_{n \geq 0} H^{-n}(V_\varepsilon \setminus \{z_3 = 0\})$. Let $\gamma: S^1 \rightarrow U^+$ be a closed curve with $K := \gamma(S^1)$. There exists $n \in \mathbb{N}$ such as $K \subset H^{-n}(V_\varepsilon \setminus L)$ that is such that $H^n(K) \subset V_\varepsilon \setminus \{z_3 = 0\}$. Since $V_\varepsilon \setminus L$ is simply connected, we conclude that the curve γ is trivial in $\pi_1(U^+)$. \square

Our next step is to give a lower bound of the complex dimension of $H^1(X, \mathcal{O})$.

Proposition 2.3. $\dim_{\mathbb{C}} H^1(X, \mathcal{O}) \geq 1$.

Proof: We shall show that $\dim H^1(X, \mathcal{O}^*)^0 \geq 1$. To do so, we construct a non trivial twisted holomorphic line bundle. We denote by \tilde{X} the universal cover of X and by \tilde{H} the generator of the fundamental group $\pi_1(X)$ induced by the initial automorphism H . Let us consider for $\lambda \neq 1$ the map:

$$\begin{aligned} \tilde{X} \times \mathbb{C} &\longrightarrow \tilde{X} \times \mathbb{C} \\ (x, z) &\longmapsto (\tilde{H}(x), \lambda z). \end{aligned}$$

The projection onto the first component gives

$$\tilde{X} \times \mathbb{C}/\mathbb{Z} =: L_\lambda \longrightarrow \tilde{X}/\mathbb{Z} = X.$$

The bundle L_λ is a flat line bundle and therefore belongs to $H^1(X, \mathcal{O}^*)^0$, since $\pi_1(X) = \mathbb{Z}$. Suppose that there exists a non-trivial section $\sigma \in H^0(X, L_\lambda)$. The section σ induces a holomorphic function $f: \tilde{X} \rightarrow \mathbb{C}$ such that $f(H(x)) = \lambda f(x)$.

We know that $X = U^+ \cup \tilde{D}/\langle \tilde{H} \rangle_{\mathbb{Z}} = \tilde{X}/\langle \tilde{H} \rangle_{\mathbb{Z}}$, where \tilde{D} , the inverse image of the divisor D in \tilde{X} , which is a connected infinite chain of rational surfaces. Here we have noted by $\langle \tilde{H} \rangle_{\mathbb{Z}}$ the group generated by the automorphism \tilde{H} .

The function f is necessarily equal to zero on the irreducible components of \tilde{D} , hence on the whole divisor $\tilde{D} \subset \tilde{X}$ by connexity.

In what follows we shall see that there is a surface with global spherical shell (GSS) naturally associated to the space X .

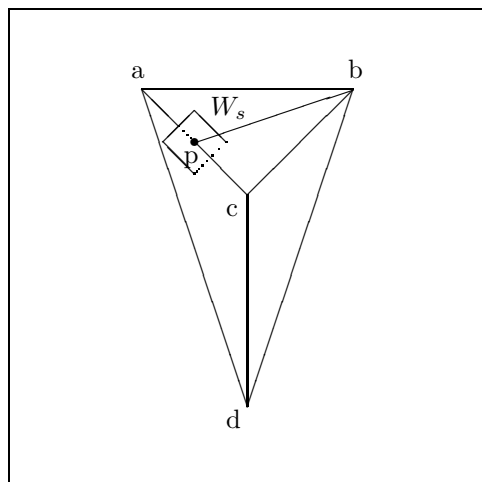
For the sake of simplicity, we consider the case $c = 0$, the other cases working analogously. Then the automorphism H is given by

$$H(x : y : z : t) = (x^2 + tz : y^2 + tx : ty : t^2).$$

The point $p = [1 : 1 : 0 : 0]$ in $l = X^+$ is a fixed point of saddle type. This is seen by an easy calculation: there are two eigenvalues equal to zero and one with absolute value strictly greater than one. At the point p , there exists locally a stable manifold noted by $W_s(p) \subset V_\varepsilon$ which is transversal to the attractor l , see [8]. Let \mathcal{V} be a neighbourhood of $0 \in \mathbb{C}^2$ immersed onto the stable manifold. This gives rise to the commutative diagram:

$$\begin{array}{ccc} (\mathcal{V}, 0) & \xrightarrow{\varphi} & (\mathcal{V}, 0) \\ \downarrow \iota & & \downarrow \iota \\ W_s(p) & \xrightarrow{H} & W_s(p) \end{array}$$

where ι denotes the immersion and φ the induced selfmapping of $(\mathcal{V}, 0)$.

FIGURE 2. Stable manifold W_s .

Coordinates:

$$a = [1 : 0 : 0 : 0]$$

$$b = [0 : 0 : 1 : 0]$$

$$c = [0 : 1 : 0 : 0]$$

$$p = [1 : 1 : 0 : 0]$$

$$e = [0 : 0 : 0 : 1].$$

Let $C(\varphi) = \iota^{-1}(\iota(\mathcal{V}) \cap L)$ (where $L = \{t = 0\}$ is the hyperplane at infinity) be the critical set of φ .

The restriction of φ to $\mathcal{V} \setminus C(\varphi)$ is injective hence biholomorphic onto its image. Consequently, φ is a strict germ of topological degree 1. According to Ch. Favre [4], φ is a Dloussky germ and thus defines a compact complex surface S with global spherical shell.

Now the holomorphic function f induces a holomorphic function $h := \iota \circ f$ on \mathcal{V} with $h \circ \varphi = \lambda h$, which is zero on $C(\varphi)$. Since $\lambda \neq 1$, the function h is identically zero on \mathcal{V} . This follows from the well-known fact that holomorphically non-trivial flat line bundles on surfaces with GSS have no non-trivial holomorphic sections, see e.g. [3].

We consider the increasing union

$$Y := \bigcup_{n \geq 0} H^{-n}(W_s(p) \setminus L) \subset U^+,$$

which is a 2-dimensional connected immersed submanifold of U^+ and which is, as an abstract manifold, biholomorphic to the complement of the maximal divisor in the universal covering \tilde{S} of the above mentioned compact surface with GSS. Its topological closure in $\mathbb{P}_3(\mathbb{C})$ contains all the projective lines of the form $\{[x : y : t : 0] \mid t \in \mathbb{C}\}$, where $x/y \in S^1$ is a 2^m -root of unity for some $m \in \mathbb{N}$. Therefore the quotient $S' := Y/\langle H \rangle_{\mathbb{Z}} \subset X$ is not contained in any closed complex hypersurface of X . Since the function h is identically zero, this implies that the section $\sigma \in H^0(X, L_\lambda)$ vanishes identically, a contradiction to our assumption. We have proved

$$\dim H^1(X, \mathcal{O}) \geq 1. \quad \square$$

Now we show that there are no meromorphic functions on X .

Proposition 2.4. *The algebraic dimension of the space X is equal to zero.*

Proof: We consider again $S' := Y/\langle H \rangle_{\mathbb{Z}} \subset X$ which is, as an abstract manifold biholomorphic to the complement of the maximal divisor in the GSS surface S . Let us suppose that there exists a meromorphic function f on X . Then f is constant on S' according to the properties of surfaces with global spherical shell. Since S' is not contained in any hypersurface in X , the function f is constant on X . Therefore, the algebraic dimension $a(X)$ is equal to zero. \square

3. Regular quadratic automorphisms allowing the construction

The example with which we have worked for the moment is just a particular case of a regular quadratic automorphism of \mathbb{C}^3 having a projective line as an attractor at infinity; such automorphisms belong to the fourth and fifth class in the classification of Fornæss-Wu. We shall give their list.

Those of the fourth class for which we can construct a compact complex space with global shell isomorphic to B are of the form:

$$H_4(x, y, z) = \begin{cases} x^2 + \alpha xy + \beta y^2 + \delta y + \gamma + az \\ y^2 + \nu + x \\ y \end{cases}$$

the coefficients are chosen arbitrarily with $a \neq 0$.

The inverse automorphism is of the form

$$H_4^{-1}(x, y, z) = \begin{cases} y - \nu - z^2 \\ z \\ \frac{1}{a}(x - P(y, z)), \quad P \text{ having } \deg_z(P) = 4. \end{cases}$$

These automorphisms admit as set of indeterminacy the point

$$I^+ = [0 : 0 : 1 : 0];$$

the inverse automorphisms the set $I^- = \{z = t = 0\}$; we check immediately that $I^+ \cap I^- = \emptyset$.

By the action of H_4 , the hyperplane at infinity $\{t = 0\}$ minus I^+ is mapped to the \mathbb{P}^1 attractor $X^+ = [x : y : 0 : 0]$.

In order to get the topological closure of the graph in $\mathbb{P}_3 \times \mathbb{P}_3$ one has to add the two divisors

$$\begin{aligned} D_1 := & \left\{ [z_0 : z_1 : z_2 : 0], [z_0^2 + \alpha z_0 z_1 + \beta z_1^2 : z_1^2 : 0 : 0] \right\} \\ & \cup \left\{ [0 : 0 : 1 : 0], [y_0 : y_1 : 0 : 0] \right\} \\ D_2 := & \left\{ [0 : 0 : 1 : 0], [y_0 : y_1 : y_2 : 0] \right\}. \end{aligned}$$

As for the automorphisms of the fifth class, they are of the form:

$$H_5(x, y, z) = \begin{cases} y^2 + \alpha xy + \beta x^2 + \delta x + \gamma + az \\ x^2 + \theta + y \\ x \end{cases}$$

the constants are chosen arbitrarily with $a \neq 0$. The inverse automorphism is of the form

$$H_5^{-1}(x, y, z) = \begin{cases} z \\ y - \theta - z^2 \\ \frac{1}{a}(x - P(y, z)), \quad P \text{ having } \deg_z(P) = 4. \end{cases}$$

These automorphisms admit as the indetermination set the point

$$I^+ = [0 : 0 : 1 : 0];$$

as for the inverse automorphisms the set $I^- = \{z = t = 0\}$; it is again clear that $I^+ \cap I^- = \emptyset$ and that H_5 maps the hyperplane at infinity $\{t = 0\}$ minus I^+ to the \mathbb{P}^1 attractor $X^+ = [x : y : 0 : 0]$.

Furthermore, for the topological closure of the graph in $\mathbb{P}_3 \times \mathbb{P}_3$ one adds

$$\begin{aligned} D_1 &:= \left\{ [z_0 : z_1 : z_2 : 0], [z_1^2 + \alpha z_0 z_1 + \beta z_0^2 : z_0 z_2 : 0 : 0] \right\} \\ &\quad \cup \left\{ [0 : 0 : 1 : 0], [y_0 : y_1 : 0 : 0] \right\} \\ D_2 &:= \left\{ [0 : 0 : 1 : 0], [y_0 : y_1 : y_2 : 0] \right\}. \end{aligned}$$

4. Conclusion

We gather the different results in:

Theorem 4.1. *The regular quadratic automorphisms of \mathbb{C}^3 in the list of Fornæss-Wu [5] from which one can construct a compact complex three-dimensional space of class L with a global spherical shell isomorphic to B are:*

(1) *In the fourth class, those of the form*

$$H_4(x, y, z) = \begin{cases} x^2 + \alpha xy + \beta y^2 + \delta y + \gamma + az \\ y^2 + \nu + x, \\ y. \end{cases} \quad a \neq 0$$

(2) *In the fifth class, those of the form*

$$H_5(x, y, z) = \begin{cases} y^2 + \alpha xy + \beta x^2 + \delta x + \gamma + az \\ x^2 + \theta + y, \\ x. \end{cases} \quad a \neq 0$$

The space X has the following properties:

- *the fundamental group is $\pi_1(X) = \mathbb{Z}$,*
- *the fundamental group of the attraction basin in \mathbb{C}^3 is: $\pi_1(U^+) = 1$,*
- *$\dim H^1(X, \mathcal{O}) \geq 1$,*
- *the algebraic dimension is equal to zero: $a(X) = 0$.*

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